Approximation Algorithms
Problem set #3

The assignment is due Monday, May 22. Please, submit your work via Canvas.

Problem 1 Consider the following problem. We are given a graph $G = (V, E)$ and two terminals s and t. Each edge $e \in E$ has a length $l(e)$ and cost $c(e)$ which are also given to us. Our goal is to find the cheapest path from s to t of length at most M. Design a $(1 + \varepsilon)$ approximation algorithm (PTAS) for the problem.

> **Hint 1:** In this problem, you don't need to use linear programming or metric embeddings.
> **Hint 2:** Round all costs to the multiples of some number and then use dynamic programming.
> **Hint 3:** You may give an algorithm that only works for Directed Acyclic Graphs (DAGs). The general case is optional.

Problem 2 Suppose that we need to deliver packages from a warehouse s to a set of clients X. We have one truck that can carry at most W pounds at a time. At every trip, it leaves the warehouse with several packages of weight at most W, delivers them to the clients, and then gets back to the warehouse. To serve all clients, the truck can make several trips. We are given the weights of all packages (the weight of the package for a client x is w_x) and the set of all distances. The distance from the warehouse s to a client x is $d(w, x)$. The distance between clients x and y is $d(x, y)$. The distance function d is symmetric and satisfies the triangle inequality, and thus $(X \cup \{s\}, d)$ is a metric space. Our goal is to find the shortest route for the truck.

Give a $O(\log n)$ approximation algorithm.

a. Design a constant factor approximation algorithm for a tree metric.
Hint: How many times does the optimal route goes along an edge (u, v)? Give a lower bound depending on the total weight of the packages that should be delivered to the clients located in the subtree rooted at v.
b. Give a $O(\log n)$ approximation algorithm for a general metric space.
Hint: Embed the metric space into a distribution of dominating trees.

Problem 3 We say that a metric space (X, d) embeds into L_1 isometrically if there exists a distribution of cuts $(S, X \setminus S)$ of X and a number α such that for all $x, y \in X$:

$$d(x, y) = \alpha \Pr(x, y \text{ are separated by the cut } (S, X \setminus S))$$

Prove that the following metric spaces embed into L_1 isometrically.

a. n points with $d(x, y) = 1$ for every $x \neq y$.
b. Any metric space on three points.
c. Square $\{x, y, z, w\}$ with $d(x, y) = d(y, z) = d(z, w) = d(w, x) = 1$; $d(x, z) = d(y, w) = \sqrt{2}$.
d. (optional problem) Any hypercube $\{0,1\}^n$ in n-dimensional Euclidian space with the standard Euclidian distance.