Approximation Algorithms
Problem set #2

The assignment is due Wednesday, May 10. Please, submit your work via Canvas.

Problem 1 (LP Duality).

1. Write dual LPs for the following linear programs:
   a. **minimize** $4x_1 + 12x_2 + 2x_3$
      subject to
      $x_1 + 5x_2 + 7x_3 \geq 20$
      $x_1 + 3x_2 + 2x_3 \geq 11$
      $2x_1 + x_2 \geq 16$
      $x_2 + x_3 \geq 4$
      $x_1, x_2, x_3 \geq 0$

   b. **maximize** $4x + y + z$
      subject to
      $x + 2y \leq 4$
      $5x + 2y \leq 7$
      $2x + z \leq 8$
      $x, y, z \geq 0$

Solve these linear programs and their duals using an LP solve (see page 3 for details). Present the results. Prove that the solutions you obtain are indeed optimal.

Problem 2 (Integrality Gap Example for Vertex Cover) The goal of this exercise is to construct an integrality gap example for the Vertex Cover problem. Let G be a complete graph on n vertices. Show that the optimum LP value for the standard LP relaxation is $n/2$. Prove that the LP integrality gap for Vertex Cover is at least $2 - o(1)$ as $n \to \infty$.

Problem 3 (Integrality Gap Example for Set Cover). The goal of this exercise is to construct an integrality gap example for Set Cover. Fix parameters n – the number of elements in the universe; $L = \lceil \log n \rceil$; $m = \lceil \sqrt{n} \rceil$ – the total number of sets; and $p = C/L$ – the probability of including an element $v \in \{1, \dots, n\}$ into a set (for some constant C).

Now, consider a collection of m random sets $S_1, \dots, S_m$, where every set S_i is a subset of $\{1, \dots, n\}$ containing each number $v \in \{1, \dots, n\}$ with probability p. All sets S_i are chosen independently. Use the following steps to prove that the integrality gap of the Set Cover instance $(S_1, .. S_m)$ is $\Omega(\log n)$.

1. Obtain a bound on the probability that an element v is covered with less than $mp/2$ sets S_i (hint: use the Chernoff bound).
2. Using item 1, prove that with a high probability over the choice of the sets S_i the following statement holds: Each element $v \in \{1, \dots, n\}$ is covered by at least $mp/2$ sets.

3. Show that there exists an LP solution of cost at most $O(1/p)$ (with high probability). Denote the cost of solution by LP^*.
4. Prove that sets S_i cover the entire set $\{1, \dots, n\}$ (with high probability).
5. Fix a subset of indices $I \subset \{1, \dots, m\}$ of size at most $c\ LP^* \log n$ (for some constant c).
    a. Show that for every $v \in \{1, \dots, n\}$, the probability that v is covered by S_i with $i \in I$ (i.e., $v \in \bigcup_{i \in I} S_i$) is at most $1 - n^{c'}$ for some c'.
    b. Derive an upper bound on the probability that every element $v \in \{1, \dots, n\}$ is covered by S_i with $i \in I$.
6. Estimate the number of subsets $I \subset \{1, \dots, m\}$ of size at most $c\ LP^* \log n$.
7. Using items 5 and 6, prove that every solution to the Set Cover instance contains at least $c\ LP^* \log n$ sets (with high probability).
8. Conclude that the LP integrality gap of Set Cover is $\Theta(\log n)$.

Instructions on how to solve a linear program numerically

1. Write your problem in the LP CPLEX file format. Here is an example:

   LP: maximize $x + y$

   subject to

   $$x + 2y \leq 4$$
   $$3x + 2y \leq 7$$
   $$x, y \geq 0$$

   LP File:

```
\comments start with “\”

Maximize x + y


Subject to
x + 2 y <= 4
3 x + 2 y <= 7

Bounds
0 <= x
0 <= y

End
```

2. Submit your LP file to an LP Solver.

   a. Go to the NEOS website: https://neos-server.org/neos/solvers/index.html
   b. Pick Linear Programing → Gurobi → LP Input
      (you can also try CPLEX or MOSEK)
   c. Upload your LP file and check the “Return .sol file” box.
   d. Submit your job and wait a few minutes for the results.

3. You should get a solution in the following format:

```
Solved in 2 iterations and 0.00 seconds

Optimal objective: 2.75
********** Begin .sol file *************
# Objective value = 2.75
x 1.5
y 1.25
********** End .sol file ***************
```