Therefore, $\mathbb{E}[\mu(S \cap A_u) \mid u \in S] \le \mu(A_u)/m \le 1/m$, and, by Markov's inequality,

$$\Pr\left(\mu(S \cap A_u) \ge \varepsilon\rho/2 \ \mid \ u \in S\right) \le \frac{\mathbb{E}[\mu(S \cap A_u) \mid u \in S]}{\varepsilon\rho/2} \le \frac{2}{m\varepsilon\rho} = \delta.$$

The last equality holds since $m = 2/(\delta\varepsilon\rho)$. We plug this bound in (15) and get the desired inequality,

$$\Pr(u \in S') \ge \alpha\|\bar{u}\|^2 \cdot (1 - \delta).$$

We now prove Lemma 5.2.

Lemma 5.2. *For every $u \in V$ such that $\bar{u} \neq 0$, $\mu(B_u) \le (1 + \varepsilon/2)\rho$.*

Proof. To prove the lemma, we first derive a lower bound on $\langle \bar{u}, \bar{v} \rangle$ for points $v \in B_u$ and an upper bound on $\sum_{v \in B_u} \langle \bar{u}, \bar{v} \rangle \mu(v)$; combining the bounds we get an upper bound for $\mu(B_u)$. If $v \in B_u$, then by the definition of B_u and by inequality (13), we have

$$\langle \bar{u}, \bar{v} \rangle = \|\bar{u}\|^2 + \underbrace{(\|\bar{v}\|^2 - \langle \bar{u}, \bar{v} \rangle)}_{\ge 0} - \|\bar{u} - \bar{v}\|^2 \ge (1 - \beta)\|\bar{u}\|^2.$$

From constraints (11) and (13),

$$\sum_{v \in B_u} \langle \bar{u}, \bar{v} \rangle \mu(v) \le \sum_{v \in V} \langle \bar{u}, \bar{v} \rangle \mu(v) \le \rho\|\bar{u}\|^2.$$

Thus,

$$\mu(B_u) = \sum_{v \in B_u} \mu(v) \le \sum_{v \in B_u} \mu(v) \cdot \frac{\langle \bar{u}, \bar{v} \rangle}{(1 - \beta)\|\bar{u}\|^2} \le \frac{\rho\|\bar{u}\|^2}{(1 - \beta)\|\bar{u}\|^2}$$
$$\le (1 + [illegible])\rho = (1 + \varepsilon/[illegible]$$

□

Acknowledgement

The authors thank the anonymous referee for valuable comments and suggestions.

References

A. Amir, J. Ficler, R. Krauthgamer, L. Roditty, and O. Sar Shalom. Multiply Balanced k-Partitioning. LATIN 2014.

N. Bansal, U. Feige, R. Krauthgamer, K. Makarychev, V. Nagarajan, J. Naor, and R. Schwartz. Min-max Graph Partitioning and Small Set Expansion. SIAM Journal on Computing 43.2 (2014): 872-904.

E. Chlamtac, K. Makarychev, and Y. Makarychev. How to Play Unique Games Using Embeddings. FOCS 2006.

R. Krauthgamer, J. Naor, and R. Schwartz. Partitioning graphs into balanced components. SODA 2009.

R. Krauthgamer, J. Naor, R. Schwartz, and K. Talwar. Non-Uniform Graph Partitioning. SODA 2014.

A. Louis and K. Makarychev. Approximation Algorithm for Sparsest k-Partitioning. SODA 2014.

H. Räcke. Optimal hierarchical decompositions for congestion minimization in networks. STOC 2008.