Approximation Algorithms for Unique Games via Orthogonal Separators

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Lecture notes are based on the papers [CMM06a, CMM06b, LM14].

1 Unique Games

In these lecture notes, we define the Unique Games problem and describe an approximation algorithm for it.

Definition 1.1 (Unique Games)**.** *Given a constraint graph $G = (V, E)$ and a set of permutations π_{uv} on $[k] = \{1, \dots, k\}$ (for all edges (u, v)), assign a value (label) x_u from the set of labels $[k] = \{1, \dots, k\}$ to each vertex u so as to satisfy the maximum number of constraints of the form $\pi_{uv}(x_u) = x_v$.*

Observe that if the instance is completely satisfiable, then it is very easy to find a solution that satisfies all constraints: We simply need to guess the label for one vertex, and then propagate the values to all other vertices in the graph. However, even if a very small fraction of all constraints is violated, then it is very hard to find a good solution. Khot [Kho02] conjectured that if the optimal solution satisfies 99% of all constraints, then it is $\mathcal{NP}$-hard to find a solution satisfying even a 1% of all constraints. The conjecture is known as Khot's Unique Games conjecture. We state it formally below.

Definition 1.2 (Unique Games Conjecture [Kho02])**.** *For every positive ε and δ, there exists a k such that given an instance of Unique Games with k labels, it is $\mathcal{NP}$-hard to distinguish between the following two cases:*

- *There exists a solution satisfying $(1 - \varepsilon)$ fraction of all constraints.*
- *Every assignment satisfies at most δ fraction of all constraints.*

It is unknown whether the conjecture is true or false. The best approximation algorithms for Unique Games find solutions satisfying $1 - O(\sqrt{\varepsilon \log k})$ and $1 - O(\varepsilon\sqrt{\log n \log k})$ fraction of all constraints, if the optimal solution satisfies $1 - \varepsilon$ fraction of all constraints.

Theorem 1.3 (Charikar, Makarychev, Makarychev [CMM06a])**.** *There exists an approximation algorithm that given a Unique Games instance satisfying $(1 - \varepsilon)$ fraction of all constraints, finds a solution satisfying $1 - O(\sqrt{\varepsilon \log k})$ fraction of all constraints.*

Theorem 1.4 (Chlamtáč, Makarychev, Makarychev [CMM06b])**.** *There exists an approximation algorithm that given a Unique Games instance satisfying $(1 - \varepsilon)$ fraction of all constraints, finds a solution satisfying $1 - O(\varepsilon\sqrt{\log n \log k})$ fraction of all constraints.*

Note that the approximation of Theorem 1.3 cannot be improved assuming the Unique Games conjecture is true [KKMO07]. We prove Theorem 1.3 using the technique of orthogonal separators from [CMM06b]. Below, we denote the number of *violated* constraints by OPT. We assume that $OPT \leq \varepsilon|E|$, and show how to find a solution violating at most $O(\sqrt{\varepsilon \log k})\,|E|$ constraints.

In the next section, we present a standard SDP relaxation for Unique Games (without ℓ_2^2 triangle inequalities). Then, in Section 3, we introduce a technique of orthogonal separators. However, we postpone the proof of existence of orthogonal separators to Section 5. In Section 4, we present the approximation algorithm and prove Theorem 1.3. Finally, in Section 6, we give some useful bounds on the Gaussian distribution.

2 SDP Relaxation

In the SDP relaxation, we have a vector $\bar{u}_i$ for every vertex $u \in V$ and label $i \in [k]$. In the *intended integral* solution, the vector $\bar{u}_i$ is the indicator of the event *"the vertex u has the label i"*. That is, if x_u^* is the optimal labeling, then the corresponding integral solution is as follows:

$$\bar{u}_i^* = \begin{cases} 1, & \text{if } x_u^* = i; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that if a constraint (u, v) is satisfied then $\bar{u}_i^* = \bar{v}_{\pi_{uv}(i)}^*$ for all i. If the constraint is violated, then $\bar{u}_i^* = \bar{v}_{\pi_{uv}(i)}^* = 0$ for all but exactly two i's: $\bar{u}_{x_u}^* = 1$, but $\bar{v}_{\pi_{uv}(x_u)}^* = 0$; and $\bar{v}_{x_v}^* = 1$, but $\bar{u}_{\pi_{uv}^{-1}(x_v)}^* = 0$. Thus,

$$\frac{1}{2} \sum_{i \in [k]} \|\bar{u}_i^* - \bar{v}_{\pi_{uv}(i)}^*\|^2 = \begin{cases} 0, & \text{if assignment } x_u^* \text{ satisfies constraint } (u, v); \\ 1, & \text{if assignment } x_u^* \text{ violates constraint } (u, v). \end{cases}$$

Therefore, the number of violated constraints equals

$$\frac{1}{2} \sum_{(u,v) \in E} \sum_{i \in [k]} \|\bar{u}_i^* - \bar{v}_{\pi_{uv}(i)}^*\|^2. \tag{1}$$

Our goal is to minimize this expression. Note that for a fixed vertex u, one and only one $\bar{u}_i^*$ equals 1. Hence,

- $\langle \bar{u}_i^*, \bar{u}_j^* \rangle = 0$, if $i \neq j$; and
- $\sum_i \|\bar{u}_i^*\|^2 = 1$.

We now write the SDP relaxation.

$$\min \quad \frac{1}{2} \sum_{(u,v) \in E} \sum_{i \in [k]} \|\bar{u}_i - \bar{v}_{\pi_{uv}(i)}\|^2$$

$$\langle \bar{u}_i, \bar{u}_j \rangle = 0 \qquad \text{for all } u \in V \text{ and } i \neq j$$

$$\sum_{i \in [k]} \|\bar{u}_i\|^2 = 1 \qquad \text{for all } u \in V$$

This is a relaxation, since for $\bar{u}_i = \bar{u}_i^*$, the SDP value equals the number of violated constraints (see (1)); and $\bar{u}_i^*$ is a feasible solution for the SDP. Usually, such relaxations contain an extra constraints – the ℓ_2^2 triangle inequality. But we will note use it here.

3 Orthogonal Separators – Overview

Let X be a set of vectors in ℓ_2 of length at most 1. We say that a distribution over subsets of X is an m-orthogonal separator of X with ℓ_2 distortion D, probability scale $\alpha > 0$ and separation threshold $\beta < 1$, if the following conditions hold for $S \subset X$ chosen according to this distribution:

1. For all $\bar{u} \in X$, $\Pr(\bar{u} \in S) = \alpha \|\bar{u}\|^2$.

2. For all $\bar{u}, \bar{v} \in X$ with $\langle \bar{u}, \bar{v} \rangle \leq \beta \max(\|\bar{u}\|^2, \|\bar{v}\|^2)$,

$$\Pr(\bar{u} \in S \text{ and } \bar{v} \in S) \leq \frac{\alpha \min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m}.$$

3. For all $\bar{u}, \bar{v} \in X$,

$$\Pr(I_S(\bar{u}) \neq I_S(\bar{v})) \leq \alpha D \|\bar{u} - \bar{v}\| \cdot \min(\|\bar{u}\|, \|\bar{v}\|) + \alpha \big| \|\bar{u}\|^2 - \|\bar{v}\|^2 \big|,$$

   where I_S is the indicator of the set S i.e. $I_S(\bar{u}) = 1$, if $\bar{u} \in S$; $I_S(\bar{u}) = 0$, if $\bar{u} \notin S$.

In most cases, it is convenient to use a slightly weaker (but simpler) bound on $\Pr(I_S(\bar{u}) \neq I_S(\bar{v}))$.

$3'$. For all $\bar{u}, \bar{v} \in X$,

$$\Pr(I_S(\bar{u}) \neq I_S(\bar{v})) \leq \alpha D \|\bar{u} - \bar{v}\| \cdot \max(\|\bar{u}\|, \|\bar{v}\|),$$

The property ($3'$) follows from (3), since

$$\big| \|\bar{u}\|^2 - \|\bar{v}\|^2 \big| = \big| \|\bar{u}\| - \|\bar{v}\| \big| \cdot (\|\bar{u}\| + \|\bar{v}\|) \leq \|\bar{u} - \bar{v}\| \cdot 2 \max(\|\bar{u}\|, \|\bar{v}\|).$$

The last inequality follows from the (regular) triangle inequality for vectors $\bar{u}$, $\bar{v}$ and $(\bar{u} - \bar{v})$.

Our algorithm for Unique Games relies on the following theorem.

Theorem 3.1 (Chlamtac, Makarychev, Makarychev [CMM06b]). *There exists a polynomial-time randomized algorithm that given a set of vectors X in the unit ball and parameter m, generates a m-orthogonal separator with ℓ_2 distortion $D = O\left(\sqrt{\log m}\right)$, probability scale $\alpha \geq poly(1/m)$ and separation threshold $\beta = 0$.*

4 Approximation Algorithm

We are now ready to present the approximation algorithm.

Input: An instance of Unique Games.
Output: Assignment of labels to vertices.

1. Solve the SDP. Let $X = \{\bar{u}_i : u \in V, i \in [k]\}$.

2. Mark all vertices as unprocessed.

3. while (there are unprocessed vertices)

   (a) Produce an m-orthogonal separator $S \subset X$ with distortion D and probability scale α as in Theorem 3.1, where $m = 4k$ and $D = O(\sqrt{\log n \log m})$.

   (b) For all unprocessed vertices u :

   - Let $S_u = \{i : u_i \in S\}$.
   - If S_u contains exactly one element i, then assign the label i to u, and mark the vertex u as processed.

4. If the algorithm performs more than n/α iterations, assign arbitrary values to any remaining vertices (note that $\alpha \geq 1/poly(k)$).

Lemma 4.1. *The algorithm satisfies the constraint between vertices u and v with probability $1-O(D\sqrt{\varepsilon_{uv}})$, where ε_{uv} is the SDP contribution of the term corresponding to the edge (u,v):*

$$\varepsilon_{uv} = \frac{1}{2}\sum_{i=1}^{k} \|\bar{u}_i - \bar{v}_{\pi_{uv}(i)}\|^2.$$

Proof. If $D\sqrt{\varepsilon_{uv}} \geq 1/8$, then the statement holds trivially, so we assume that $D\sqrt{\varepsilon_{uv}} < 1/8$. For the sake of analysis we also assume that π_{uv} is the identity permutation (we can just rename the labels of the vertex v, this clearly does not affect the execution of the algorithm).

At the end of an iteration in which one of the vertices u or v assigned a value we mark the constraint as satisfied or not: the constraint is satisfied, if the the same label i is assigned to the vertices u and v; otherwise, the constraint is not satisfied (here we conservatively count the number of satisfied constraints: a constraint marked as not satisfied in the analysis may potentially be satisfied in the future).

Consider one iteration of the algorithm. There are three possible cases:

1. Both sets S_u and S_v are equal and contain only one element, then the constraint is satisfied.

2. The sets S_u and S_v are equal, but contain more than one or none elements, then no values are assigned at this iteration to u and v.

3. The sets S_u and S_v are not equal, then the constraint is not satisfied (a conservative assumption).

Let us estimate the probabilities of each of these events. Using the fact that for all $i \neq j$ the vectors $\bar{u}_i$ and $\bar{u}_j$ are orthogonal, and the first and second properties of orthogonal separators we get (below α is the probability scale): for a fixed i,

$$\begin{aligned}
\Pr\left(|S_u| = 1;\ i \in S_u\right) &= \Pr\left(i \in S_u\right) - \Pr\left(i \in S_u \text{ and } j \in S_u \text{ for some } j \neq i\right) \\
&\geq \Pr\left(i \in S_u\right) - \sum_{\substack{j\in[k]\\ j\neq i}} \Pr\left(i, j \in S_u\right) \\
&\geq \alpha\|\bar{u}_i\|^2 - \sum_{\substack{j\in[k]\\ j\neq i}} \frac{\alpha \min(\|\bar{u}_i\|^2, \|\bar{u}_j\|^2)}{4k} \\
&\geq \alpha\|\bar{u}_i\|^2 - \frac{\alpha}{4k}\sum_{\substack{j\in[k]\\ j\neq i}} \|\bar{u}_j\|^2 \\
&\geq \alpha\|\bar{u}_i\|^2 - \frac{\alpha}{4k}.
\end{aligned}$$

Here we used that $\sum_{j\in[k]} \|\bar{u}_j\|^2 = 1$. Then,

$$\Pr\left(|S_u| = 1\right) = \sum_{i\in[k]} \Pr\left(|S_u| = 1;\ i \in S_u\right) \geq \sum_{i\in[k]} \left[\alpha\|\bar{u}_i\|^2 - \frac{\alpha}{4k}\right] \geq 3/4\,\alpha.$$

The probability that the constraint is not satisfied is at most

$$\Pr\left(S_u \neq S_v\right) \leq \sum_{i\in[k]} \Pr\left(I_S(u_i) \neq I_S(v_i)\right).$$

By the third property of orthogonal separators (see property $(3')$):

$$\Pr\left(S_u \neq S_v\right) \leq \alpha D \sum_{i\in[k]} \|\bar{u}_i - \bar{v}_i\| \cdot \max(\|\bar{u}_i\|, \|\bar{v}_i\|).$$

By Cauchy-Schwarz,

$$\begin{aligned}\Pr\left(S_u \neq S_v\right) &\leq \alpha D \sqrt{\sum_{i\in[k]} \|\bar{u}_i - \bar{v}_i\|^2} \cdot \sqrt{\sum_{i\in[k]} \max(\|\bar{u}_i\|^2, \|\bar{v}_i\|^2)} \\ &\leq \alpha D \sqrt{\sum_{i\in[k]} 2\varepsilon_{uv}} \cdot \underbrace{\sqrt{\sum_{i\in[k]} \|\bar{u}_i\|^2 + \|\bar{v}_i\|^2}}_{=\sqrt{2}} \\ &= 2\alpha D\sqrt{\varepsilon_{uv}}.\end{aligned}$$

Finally, the probability of satisfying the constraint is at least

$$\Pr\left(|S_u| = 1 \text{ and } S_u = S_v\right) \geq \frac{3}{4}\alpha - 2\alpha D\sqrt{\varepsilon_{uv}} \geq \frac{1}{2}\,\alpha.$$

Since the algorithm performs n/α iterations, the probability that it does not assign any value to u or v before step 4 is exponentially small. At each iteration the probability of failure is at most $O(D\sqrt{\varepsilon_{uv}})$ times the probability of success, thus the probability that the constraint is not satisfied is $O(D\sqrt{\varepsilon_{uv}})$. □

We now show that the approximation algorithm satisfies $1 - O(\sqrt{\varepsilon \log k})$ fraction of all constraints.

Proof of Theorem 1.3. By Lemma 4.1, the expected number of unsatisfied constraints is equal to

$$\sum_{(u,v)\in E} O(\sqrt{\varepsilon_{uv} \log k}).$$

By Jensen's inequality for the function $t \mapsto \sqrt{t}$,

$$\frac{1}{|E|} \sum_{(u,v)\in E} \sqrt{\varepsilon_{uv} \log k} \leq \sqrt{\frac{1}{|E|} \sum_{(u,v)\in E} \varepsilon_{uv} \log k} = \sqrt{\frac{SDP}{|E|} \log k}.$$

Here, $SDP = \sum_{(u,v)\in E} \varepsilon_{uv}$ denotes the SDP value. If $OPT \leq \varepsilon|E|$, then $SDP \leq OPT \leq \varepsilon|E|$. Hence, the expected cost of solution is upper bounded by $O(\sqrt{\varepsilon \log k})|E|$. □

5 Orthogonal Separators – Proofs

Proof of Theorem 3.1. In the proof, we denote the probability that a Gaussian $\mathcal{N}(0,1)$ random variable X is greater than a threshold t by $\bar{\Phi}(t)$. We use the following algorithm for generating m-orthogonal separators with ℓ_2 distortion: Assume w.l.o.g. that all vectors $\bar{u}$ lie in $\mathbb{R}^n$. Fix $m' = m$ and $t = \bar{\Phi}^{-1}(1/m')$ (i.e., fix t such that $\bar{\Phi}(t) = 1/m'$). Sample independently a random Gaussian n dimensional vector $g \sim \mathcal{N}(0, I)$ in $\mathbb{R}^n$ and a random number r in $[0, 1]$. Return the set

$$S = \{\bar{u} : \langle \bar{u}, g\rangle \geq t\|\bar{u}\| \text{ and } \|\bar{u}\|^2 \geq r\}.$$

We claim that S is an m-orthogonal separator with ℓ_2 distortion $O(\sqrt{\log m})$, probability scale $\alpha = 1/m'$ and $\beta = 0$. Let us verify that S satisfies the required conditions.

1. For every nonzero vector $\bar{u} \in X$, we have

$$\begin{aligned}\Pr(\bar{u} \in S) &= \Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\| \text{ and } r \leq \|\bar{u}\|^2) \\ &= \Pr(\langle \bar{u}/\|\bar{u}\|, g\rangle \geq t) \cdot \Pr(r \leq \|\bar{u}\|^2) \\ &= \|\bar{u}\|^2/m' \equiv \alpha\|\bar{u}\|^2.\end{aligned}$$

Here we used that $\langle \bar{u}/\|\bar{u}\|, g\rangle$ is distributed as $\mathcal{N}(0,1)$, since $\bar{u}/\|\bar{u}\|$ is a unit vector. If $\bar{u} = 0$, then $\Pr(r \leq \|\bar{u}\|^2) = 0$, hence $\Pr(\bar{u} \in S) = 0$.

2. For every $\bar{u}, \bar{v} \in X$ with $\langle \bar{u}, \bar{v}\rangle = 0$, we have

$$\begin{aligned}\Pr(\bar{u}, \bar{v} \in S) &= \Pr(\langle \bar{u}, g\rangle \geq t;\ \langle \bar{v}, g\rangle \geq t;\ r \leq \|\bar{u}\|^2 \text{ and } r \leq \|\bar{v}\|^2) \\ &= \Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\| \text{ and } \langle \bar{v}, g\rangle \geq t\|\bar{v}\|) \cdot \Pr(r \leq \min(\|\bar{u}\|^2, \|\bar{v}\|^2)).\end{aligned}$$

The random variables $\langle \bar{u}, g\rangle$ and $\langle \bar{v}, g\rangle$ are independent, since $\bar{u}$ and $\bar{v}$ are orthogonal vectors. Hence,

$$\Pr(\bar{u}, \bar{v} \in S) = \Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\|) \cdot \Pr(\langle \bar{v}, g\rangle \geq t\|\bar{v}\|) \cdot \Pr(r \leq \min(\|\bar{u}\|^2, \|\bar{v}\|^2)).$$

Note that $\bar{u}/\|\bar{u}\|$ is a unit vector, and $\langle \bar{u}/\|\bar{u}\|, g\rangle \sim \mathcal{N}(0,1)$. Thus,

$$\Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\|) = \Pr(\langle \bar{u}/\|\bar{u}\|, g\rangle \geq t) = 1/m'.$$

Similarly, $\Pr(\langle \bar{v}, g\rangle \geq t\|\bar{v}\|) = 1/m'$. Then, $\Pr(r \leq \min(\|\bar{u}\|^2, \|\bar{v}\|^2)) = \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$, since r is uniformly distributed in $[0,1]$. We get

$$\Pr(\bar{u}, \bar{v} \in S) = \frac{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m'^2} = \frac{\alpha \min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m}.$$

3. If $I_S(\bar{u}) \neq I_S(\bar{v})$ then either $\bar{u} \in S$ and $\bar{v} \notin S$, or $\bar{u} \notin S$ and $\bar{v} \in S$. Thus,

$$\Pr(I_S(\bar{u}) \neq I_S(\bar{v})) = \Pr(\bar{u} \in S; \bar{v} \notin S) + \Pr(\bar{u} \notin S; \bar{v} \in S).$$

We upper bound the both terms on the right hand side using the following lemma (switching $\bar{u}$ and $\bar{v}$ for the second term) and obtain the desired inequality.

Lemma 5.1. *If $\|\bar{u}\|^2 \geq \|\bar{v}\|^2$, then*

$$\Pr(\bar{u} \in S; \bar{v} \notin S) \leq \alpha D\|\bar{u} - \bar{v}\| \cdot \min(\|\bar{u}\|^2, \|\bar{v}\|^2) + \alpha\big|\|\bar{u}\| - \|\bar{v}\|\big|;$$

otherwise,

$$\Pr(\bar{u} \in S; \bar{v} \notin S) \leq \alpha D\|\bar{u} - \bar{v}\| \cdot \min(\|\bar{u}\|^2, \|\bar{v}\|^2).$$

Proof of Lemma 5.1. We have

$$\Pr(\bar{u} \in S; \bar{v} \notin S) = \Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\|;\ r \leq \|\bar{u}\|^2;\ v \notin S).$$

The event $\{\bar{v} \notin S\}$ is the union of two events $\{\langle \bar{v}, g\rangle \geq t\|\bar{v}\| \text{ and } r \leq \|\bar{v}\|^2\}$ and $\{r \geq \|\bar{v}\|^2\}$. Hence,

$$\begin{aligned}\Pr(\bar{u} \in S; \bar{v} \notin S) \leq\ & \Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\|;\ \langle \bar{v}, g\rangle < t\|\bar{v}\|;\ r \leq \min(\|\bar{u}\|^2, \|\bar{u}\|^2)) \\ & + \Pr(\langle \bar{u}, g\rangle \geq t\|\bar{u}\|;\ \|\bar{v}\|^2 \leq r \leq \|\bar{u}\|^2).\end{aligned} \tag{2}$$

Let $g_u = \langle \bar{u}/\|\bar{u}\|, g\rangle$ and $g_v = \langle \bar{v}/\|\bar{v}\|, g\rangle$. Both g_u and g_v are standard $\mathcal{N}(0,1)$ Gaussian random variables. Thus, $\Pr(g_u \geq t) = \Pr(g_v \geq t) = 1/m' = \alpha$. We rewrite (2) as follows:

$$\begin{aligned}\Pr(\bar{u} \in S; \bar{v} \notin S) &= \Pr(g_u \geq t;\ g_v < t)\Pr(r \leq \min(\|\bar{u}\|^2, \|\bar{v}\|^2)) + \Pr(g_u \geq t)\Pr(\|\bar{v}\|^2 \leq r \leq \|\bar{u}\|^2) \\ &= \Pr(g_u \geq t;\ g_v < t) \cdot \min(\|\bar{u}\|^2, \|\bar{v}\|^2) + \alpha\Pr(\|\bar{v}\|^2 \leq r \leq \|\bar{u}\|^2).\end{aligned} \tag{3}$$

To finish the proof we need to estimate $\Pr(g_u \geq t;\ g_v < t)$ and $\Pr(\|\bar{v}\|^2 \leq r \leq \|\bar{u}\|^2)$. Since r is uniformly distributed in $[0,1]$, $\Pr(\|\bar{v}\|^2 \leq r \leq \|\bar{u}\|^2) = \|\bar{u}\|^2 - \|\bar{v}\|^2$, if $\|\bar{u}\|^2 - \|\bar{v}\|^2 > 0$; and $\Pr(\|\bar{v}\|^2 \leq r \leq \|\bar{u}\|^2) = 0$, otherwise.

We use Lemma 6.2 to upper bound $\Pr(g_u \geq t;\ g_v < t)$:

$$\Pr(g_u \geq t;\ g_v < t) \leq O\big(\sqrt{1 - \mathrm{cov}(g_u, g_v)} \cdot \sqrt{\log m'}/m'\big). \tag{4}$$

The covariance of g_u and g_v equals $\mathrm{cov}(g_u, g_v) = \langle \bar{u}/\|\bar{u}\|, \bar{v}/\|\bar{v}\| \rangle$ and $\|\bar{u} - \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\langle \bar{u}, \bar{v} \rangle$. Hence,

$$\begin{aligned} 1 - \mathrm{cov}(g_u, g_v) = 1 - \frac{\|\bar{u}\|^2 + \|\bar{v}\|^2 - \|\bar{u} - \bar{v}\|^2}{2\|\bar{u}\|\,\|\bar{v}\|} &= \frac{\|\bar{u} - \bar{v}\|^2 - (\|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\,\|\bar{v}\|)}{2\|\bar{u}\|\,\|\bar{v}\|} \\ &= \frac{\|\bar{u} - \bar{v}\|^2 - (\|\bar{u}\| - \|\bar{v}\|)^2}{2\|\bar{u}\|\,\|\bar{v}\|} \leq \frac{\|\bar{u} - \bar{v}\|^2}{2\|\bar{u}\|\,\|\bar{v}\|}. \end{aligned}$$

We plug this bound in (4) and get

$$\Pr(g_u \geq t;\ g_v < t) \leq \alpha \cdot \frac{\|\bar{u} - \bar{v}\|}{\sqrt{\|\bar{u}\|\,\|\bar{v}\|}} \cdot O(\sqrt{\log m'}).$$

Now, Lemma 5.1 follows form (3). This concludes the proof of Lemma 5.1 and Theorem 3.1. □

□

6 Gaussian Distribution

In this section, we prove several useful estimates on the Gaussian distribution. Let $X \sim \mathcal{N}(0,1)$ be one dimensional Gaussian random variable. Denote the probability that $X \geq t$ by $\bar{\Phi}(t)$:

$$\bar{\Phi}(t) = \Pr(X \geq t).$$

The first lemma gives a very accurate estimate on $\bar{\Phi}(t)$ for large t.

Lemma 6.1. *For every $t > 0$,*

$$\frac{t}{\sqrt{2\pi}\,(t^2+1)} e^{-\frac{t^2}{2}} < \bar{\Phi}(t) < \frac{1}{\sqrt{2\pi}\,t} e^{-\frac{t^2}{2}}.$$

Proof. Write

$$\begin{aligned} \bar{\Phi}(t) &= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-\frac{x^2}{2}}}{x} \Bigg|_t^\infty - \int_t^\infty \frac{e^{-\frac{x^2}{2}}}{x^2} dx \right] \\ &= \frac{1}{\sqrt{2\pi} t} e^{-\frac{t^2}{2}} - \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{e^{-\frac{x^2}{2}}}{x^2} dx. \end{aligned}$$

Thus,

$$\bar{\Phi}(t) < \frac{1}{\sqrt{2\pi} t} e^{-\frac{t^2}{2}}.$$

On the other hand,

$$\frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{e^{-\frac{x^2}{2}}}{x^2} dx < \frac{1}{\sqrt{2\pi} t^2} \int_t^\infty e^{-\frac{x^2}{2}} dx = \frac{\bar{\Phi}(t)}{t^2}.$$

Hence,

$$\bar{\Phi}(t) > \frac{1}{\sqrt{2\pi}t} e^{-\frac{t^2}{2}} - \frac{\bar{\Phi}(t)}{t^2},$$

and, consequently,

$$\bar{\Phi}(t) > \frac{t}{\sqrt{2\pi}(t^2+1)} e^{-\frac{t^2}{2}}.$$

□

Lemma 6.2. *Let X and Y be Gaussian $\mathcal{N}(0,1)$ random variables with covariance* $\operatorname{cov}(X,Y) = 1 - 2\varepsilon^2$. *Pick the threshold $t > 1$ such that $\bar{\Phi}(t) = 1/m$ for $m > 3$. Then*

$$\Pr(X \geq t \textit{ and } Y \leq t) = O(\varepsilon\sqrt{\log m}/m).$$

Proof. If $\varepsilon t \geq 1$ or $\varepsilon \geq 1/2$, then we are done, since $\varepsilon\sqrt{\log m} = \Omega(\varepsilon t) = \Omega(1)$ and

$$\Pr(X \geq t \text{ and } Y \leq t) \leq \Pr(X \geq t) = \frac{1}{m}.$$

So we assume that $\varepsilon t \leq 1$ and $\varepsilon < 1/2$. Let

$$\xi = \frac{X+Y}{2\sqrt{1-\varepsilon^2}}; \qquad \eta = \frac{X-Y}{2\varepsilon}.$$

Note that ξ and η are $\mathcal{N}(0,1)$ Gaussian random variables with covariance 0. Hence, ξ and η are independent. We have

$$\Pr\big(X \geq t \text{ and } Y \leq t\big) = \Pr\big(\sqrt{1-\varepsilon^2}\,\xi + \varepsilon\eta \geq t \text{ and } \sqrt{1-\varepsilon^2}\,\xi - \varepsilon\eta \leq t\big).$$

Denote by $\mathcal{E}$ the following event:

$$\mathcal{E} = \big\{\sqrt{1-\varepsilon^2}\,\xi + \varepsilon\eta \geq t \text{ and } \sqrt{1-\varepsilon^2}\,\xi - \varepsilon\eta \leq t\big\}.$$

Then,

$$\Pr\big(X \geq t \text{ and } Y \leq t\big) = \Pr(\mathcal{E} \text{ and } \varepsilon\eta \leq t) + \Pr(\mathcal{E} \text{ and } \varepsilon\eta \geq t).$$

Observe that the second probability on the right hand side is very small. It is upper bounded by $\Pr(\varepsilon\eta \geq t)$, which, in turn, is bounded as follows:

$$\Pr(\varepsilon\eta \geq t) = \frac{1}{\sqrt{2\pi}} \int_{t/\varepsilon}^{\infty} e^{-\frac{x^2}{2}}\,dx = \bar{\Phi}(t/\varepsilon) \leq O\Big(\frac{\varepsilon\, e^{-\frac{t^2}{2\varepsilon^2}}}{t}\Big) \leq O\Big(\frac{\varepsilon\, e^{-\frac{t^2}{2}}}{t}\Big) = O(\varepsilon/m).$$

We now estimate the first probability:

$$\begin{aligned}\Pr(\mathcal{E} \text{ and } \varepsilon\eta \leq t) &= \mathbb{E}_\eta[\Pr(\mathcal{E} \text{ and } \eta \leq t/\varepsilon \mid \eta)] \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{t/\varepsilon} \Pr(\mathcal{E} \mid \eta = x)\, e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{t/\varepsilon} \Pr(\sqrt{1-\varepsilon^2}\xi \in [t-\varepsilon x, t+\varepsilon x])\, e^{-x^2/2}\,dx.\end{aligned}$$

The density of the random variable $\sqrt{1-\varepsilon^2}\,\xi$ in the interval $(t-\varepsilon x, t+\varepsilon x)$ for $x \in [0, t/\varepsilon]$ is at most

$$\frac{1}{\sqrt{2\pi(1-\varepsilon^2)}} e^{\frac{-(t-\varepsilon x)^2}{2(1-\varepsilon^2)}} \leq \frac{1}{2}\, e^{\frac{-(t-\varepsilon x)^2}{2}} \leq \frac{1}{2}\, e^{\frac{-t^2}{2}} \cdot e^{\varepsilon t x} \leq \frac{1}{2}\, e^{\frac{-t^2}{2}} \cdot e^{x},$$

here we used that $\varepsilon \geq 1/2$ and $\varepsilon t \geq 1$. Hence,

$$\Pr(t - \varepsilon x \leq \sqrt{1-\varepsilon^2}\,\xi \leq t + \varepsilon x) \leq \varepsilon x\, e^{\frac{-t^2}{2}} \cdot e^x.$$

Therefore,

$$\Pr(\mathcal{E} \text{ and } \varepsilon\eta \leq t) \leq \frac{\varepsilon\, e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} \int_0^{t/\varepsilon} x e^x \cdot e^{\frac{-x^2}{2}} dx \leq \frac{\varepsilon\, e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} \underbrace{\int_0^{\infty} x e^x \cdot e^{\frac{-x^2}{2}} dx}_{O(1)}.$$

The integral in the right hand side does not depend on any parameters, so it can be upper bounded by some constant (e.g. one can show that it is upper bounded by $2\sqrt{2\pi}$). We get

$$\Pr(\mathcal{E} \text{ and } \varepsilon\eta \leq t) \leq O(\varepsilon\, e^{\frac{-t^2}{2}}) = O(\varepsilon \cdot t\bar{\Phi}(t)) = O(\varepsilon\sqrt{\log m}/m).$$

This finishes the proof. □

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